Chapter 3

A Model for Sand Production in a Deforming Sand Matrix

As discussed in the previous chapter, sand production strongly hinges upon both hydrodynamical and geomechanical phenomena. From a mechanics point of view, modelling sand production is a challenging task since it involves capturing the whole range of material response from bulk to fluid-like behaviours. The discrete element method seems to lend itself readily to the computation of the physical phenomenon; however, this method is limited due to severe difficulties, such as, computational requirements, precise description of contact behaviour between granular particles, and the rigorous incorporation of the fluid phases. Hence, a continuum mechanics approach within homogenization mixture theory is still powerful alternative to predict the sand initiation and the volumetric sand production.

In this chapter, an advanced erosion-mechanics model is developed for the problem of sand production much in the same spirit as the work of Vardoulakis et al. (1996)[64]. One of the main features of the newly developed theoretical work is the *consistent* description of erosion in a *deforming* sand matrix. A Representative Elementary Volume (REV) comprised of three constituents (solid, fluid, and fluidized solid) is chosen upon which mass balance and particle transport equations are written. The erosion process is described by a particle generation constitutive law. The coupled non-linear governing equations are derived with principal field variables being porosity, fluidized sand concentration, pressure distributions and displacements. The geomechanical aspects of the matrix skeleton is naturally incorporated into the hydro-mechanical erosion model. It is known that in situ stresses as well as stresses induced while pumping and drilling cause the solid skeleton to deform (mostly dilate), and thus impact on its propensity to sand production. Some of the materials contained in this chapter were published in part during the development of this model, see Wan and Wang (2000[96], 2001[97]).

3.1 Motivations

There is an increasing interest to describe the multiphase flow occurring in deformable porous media. The continuum theory of mixtures is a common approach, whose basic premise is that each point in the space occupied by the mixture is occupied by a particle belonging to each constituent. Such a co-occupied space at the same time by the various constituents that comprise the mixture can only be given within the context of an approximate homogenization (Whitaker, 1973[63]). It is pertinent to mention two commonly used approaches as follows.

- Single phase continuum approach

In this method, all components of the multiphase system are supposed to be simultaneously present everywhere and to occupy the whole domain *without considering any interaction between phases.* Thus, for a continuum theory of mixtures to be applicable, the various constituents should be distributed in such a manner that each of them could in their own right be considered as a single continuum. In some local sense each constituent will obey the constitutive relation for that constituent alone, and the volume fraction of the constituents is introduced to account for the discontinuous distribution. The volume fraction is defined as the volume of each constituent per unit volume of the material.

- Multiphase approach

This approach employs the technique of local volume averaging: the system is supposed to consist of *interpenetrating continua* (each occupying only part of the space). These variables are continuous within each phase, but discontinuous over the entire domain. The balance laws of the continuum mechanics are averaged over some local representative elementary volume. The interaction among the phases can be expressed by the interfacial terms (Li et al., 1990[98]; Hiltunen, 1997[99]).

For simplicity, the impetus of this research is to build a continuum based model which focuses on the erosion mechanics aspect of sand production. Single phase continuum theory of mixtures applied to a fluid saturated porous medium is employed to explore the formation and growth of wormholes (cavities). Sand production is viewed as an inverse filtration phenomenon in which the transport of grains is initiated during fluid flow though a porous matrix whose porosity, density and permeability are constantly being modified. One of the goals is to relate oil rate enhancement to continuous sand influx in terms of strength of the solid matrix as well.

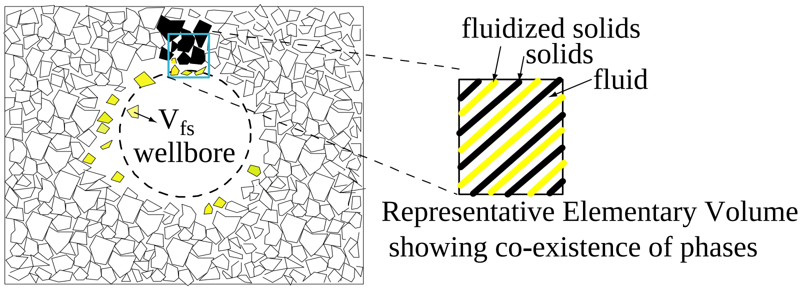


Figure 3.1: A representative elementary volume

3.2 Basic assumptions and definitions

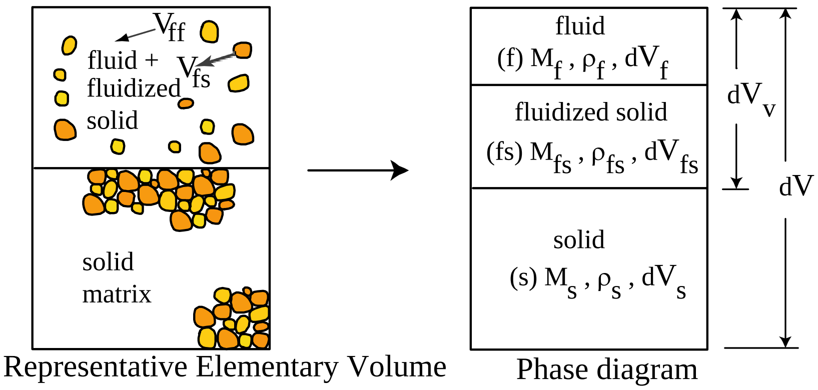
From a mechanistic viewpoint, sand production emerges as a result of an instability phenomenon occurring in a viscous fluid saturated porous medium which undergoes mechanical deformation in the presence of fluid fluxes. Despite the non-homogeneous structure of the porous medium, it is still possible to invoke a continuum theory of mixtures such that all three constituents, namely: (1) solid (s), (2) fluidized solid (fs), and (3) fluid (ff), are simultaneously present everywhere to occupy a chosen Representative Elementary Volume (REV) as shown in Figure 3.1. The size of the REV is of the order of several hundreds or thousands of pore spaces.

In order to keep the formulations tractable, the following assumptions are made:

1. Solid skeleton is described within infinitesimal strain theory and is assumed to be isotropic and homogenous.

2. The linear stress-strain relationship is not applicable, and the nonlinear relationship is used instead, i.e. nonlinear elastic or elasto-plastic).

3. The stress is defined such that compaction is *negative*.



Representative Elementary Volume Phase diagram

Figure 3.2: Partitioning REV into three distinct phases

4. The pores are completely filled with fluid and the pore fluid pressure is defined as compressive positive.
5. The fluidized particles are particles in suspension that move with fluid.
6. The densities of both solid and fluidized solid phases are equal and constant, i.e. $\rho_{fs} = \rho_s$.
7. The fluid is incompressible so that its density ρ_{ff} is constant.
8. There is no momentum transfer between solid and fluid, e.g. both fluid and fluidized particles have the same velocity, i.e. $\dot{\mathbf{u}}_{fs} = \dot{\mathbf{u}}_{ff} = \dot{\mathbf{u}}_f$.

It is pertinent to define the controlling field parameters that affect both the behaviour and interaction of each phase. These field variables are continuous as follows (see Figure 3.2).

- Porosity $\phi = \frac{dV_V}{dV}$.

- Concentration of fluidized solid $c = \frac{dV_{fs}}{dV_V}$.
- Volume discharge rate $\mathbf{v} = \frac{dV}{dSdt}$, where dV is the volume of flow through the average cross-sectional area dS in time dt.

Throughout the subsequent mathematical formulation, Einstein index notation is used in the sense that repeated indices mean summation.

3.3 Proposed mathematical model

In order to derive the governing equations, a few mathematical expressions need to be reviewed:

- Material time derivative

Given a function $e(\mathbf{x}, t)$, its material time derivative is defined by

$$\frac{De}{Dt} = \frac{\partial e}{\partial t} + \frac{\partial e}{\partial \mathbf{x}} \cdot \mathbf{v} \tag{3.1}$$

where $\mathbf{v}$ is the velocity of the moving phase boundary.

- Reynolds' transport theorem (material time derivative of a volume integral)

Consider a domain Ω of volume $\Omega(t)$ enclosed by a moving material surface $S(t)$ described by the Equation $F(x_i, t) = 0$, of area $S(t)$, and let e denote the density of the quantity, E. The total amount of E contained within $\Omega(t)$ is given by $\int_{\Omega(t)} e \, d\Omega$. The rate of change of this quantity, in the course of time, is given by $\frac{D}{Dt} \int_{U(t)} e d\Omega$, with $D(.)/Dt$ denoting the material derivative as viewed by an observer moving with

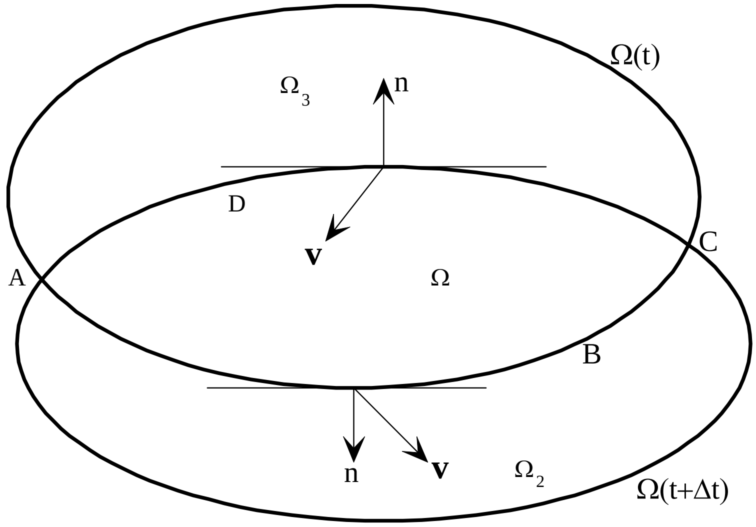


Figure 3.3: Reynolds' transport theorem

$S(t)$. Since the integral is over a varying domain, $\Omega(t)$, the order of differentiation and integration cannot be exchanged.

Figure 3.3 shows this volume at time t and at time $t + \Delta t$. By definition,

$$\begin{aligned}
& \frac{D}{Dt}\int_{\Omega(t)} e d\Omega \qquad (3.2)\\
= \; & \lim_{\Delta t \to 0}\frac{1}{\Delta t}\left\{\int_{\Omega(t+\Delta t)} e(t+\Delta t)d\Omega - \int_{\Omega(t)} e(t)d\Omega\right\}\\
= \; & \lim_{\Delta t \to 0}\frac{1}{\Delta t}\left\{\int_{\Omega} e(t+\Delta t)d\Omega - \int_{\Omega} e(t)d\Omega + \int_{\Omega_2} e(t+\Delta t)d\Omega - \int_{\Omega_3} e(t)d\Omega\right\}\\
= \; & \int_{\Omega}\frac{\partial e(t)}{\partial t}d\Omega + \lim_{\Delta t \to 0}\frac{1}{\Delta t}\left\{\int_{\Omega_2} e(t+\Delta t)d\Omega - \int_{\Omega_3} e(t)d\Omega\right\}
\end{aligned}$$

With the nomenclature of Figure 3.3, one obtains,

$$U_2 = \int_{(\Delta t)} \left(\int_{(ABC)} \mathbf{v} \cdot \mathbf{n} dS \right) dt \quad U_3 = -\int_{(\Delta t)} \left(\int_{(ADC)} \mathbf{v} \cdot \mathbf{n} dS \right) dt \tag{3.3}$$

Since $dU_2 = (\mathbf{v} \cdot \mathbf{n} dS)\, dt$ and $dU_3 = -(\mathbf{v} \cdot \mathbf{n} dS)\, dt$, Equation (3.2) can be rewritten for $\Delta t \to 0$, in the form

$$\frac{D}{Dt} \int_{U(t)} e d\Omega = \int_{\Omega_1} \frac{\partial e(t)}{\partial t} d\Omega + \int_{S(t)} e\mathbf{v} \cdot \mathbf{n} dS \tag{3.4}$$

Recalling Gauss' theorem,

$$\int_{S(t)} e\mathbf{v} \cdot \mathbf{n} dS = \int_{\Omega} \frac{\partial e\mathbf{v}}{\partial x_i} d\Omega \tag{3.5}$$

One can rewrite

$$\begin{aligned} \frac{D}{Dt} \int_{\Omega} e d\Omega &= \int_{\Omega} \frac{\partial e(t)}{\partial t} d\Omega + \int_{\Omega} \left(\frac{\partial e}{\partial x_i} \mathbf{v} + e \frac{\partial \mathbf{v}}{\partial x_i} \right) d\Omega \\ &= \int_{\Omega} \left(\frac{De}{Dt} + e \frac{\partial \mathbf{v}}{\partial x_i} \right) d\Omega = \int_{\Omega} \left(\frac{\partial e}{\partial t} + \boldsymbol{\nabla}(e\mathbf{v}) \right) d\Omega \end{aligned} \tag{3.6}$$

In words, the rate of change of the density e contained in a domain $\Omega(t)$, enclosed by a material surface $S(t)$, can be represented as the sum of two contributions:

1. the rate of the change of e integrated over the (instantaneously) fixed domain $\Omega(t)$, and
2. the net efflux of e across the (instantaneous) surface $S(t)$.

3.3.1 Mass conservation equations

Let e denote the density of a mass ρ, Equation 3.6 becomes

$$\frac{D}{Dt}\int_{\Omega}\rho d\Omega=\int_{\Omega}\left(\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}(\rho\dot{\mathbf{u}})\right)d\Omega=\int_{\Omega}\dot{m}\;d\Omega \tag{3.7}$$

where $\dot{\mathbf{u}}$ is velocity, and $\dot{m}$ is the source or sink term referring to the change in mass inside a domain $\Omega(t)$. This change is due to the variation of the density of a mass with time, e.g., the material is compressible and it becomes denser or looser.

To obtain the governing equations for the mechanics of fully saturated porous media, balance equations are written for each phase factored with its corresponding volume fraction so as to account for its discontinuous distribution. It is reminded that the volume fraction is defined as the volume of each phase per unit volume of the material. More precisely, mass conservation equations are written for each phase in terms of every parameter and state variable related to it.

solid phase (s)

The mass of the solid (M_s) phase averaged out over a REV of volume Ω can be written as

$$M_s=\int_{\Omega}(1-\phi)\,\rho_{_s}d\Omega \tag{3.8}$$

where ϕ is the porosity, $\rho_{_s}$ the density of the solid phase, and $(1-\phi)\,\rho_{_s}$ the homogenized solid density. The mass conservation requires that

$$
\begin{aligned}
\frac{DM_s}{Dt} &= \frac{D}{Dt}\int_\Omega (1-\phi)\,\rho_{_s} d\Omega \\
&= \int_\Omega \left(\frac{\partial\left[(1-\phi)\,\rho_{_s}\right]}{\partial t} + \boldsymbol{\nabla}\cdot\left[(1-\phi)\,\rho_{_s}\dot{\mathbf{u}}_s\right] \right) d\Omega = -\int_\Omega \dot{m}\; d\Omega
\end{aligned} \tag{3.9}
$$

where $\dot{\mathbf{u}}_s$ is the absolute velocity of the solid phase boundary, and the negative sign of the right hand side refers to a solid loss due to erosion since $\dot{m}$ is chosen to be the local rate of solid gain per unit volume as seen from the fluidized solid phase. Considering the arbitrariness of Ω, the local mass balance equation becomes

$$
\frac{\partial\left[(1-\phi)\,\rho_{_s}\right]}{\partial t} + \boldsymbol{\nabla}\cdot\left[(1-\phi)\,\rho_{_s}\dot{\mathbf{u}}_s\right] = -\dot{m} \tag{3.10}
$$

fluidized solid phase (fs)

The mass of fluidized solid averaged over a given REV is obtained by invoking fluidized solid concentration c, porosity ϕ, and density of the fluidized solid phase $\rho_{_{fs}}$, during homogenization, i.e.

$$
M_{_{fs}} = \int_\Omega \phi c \rho_{_{fs}} d\Omega \tag{3.11}
$$

The mass conservation of fluidized solids reduces to the following local balance equation

$$
\frac{\partial\left[\phi c \rho_{_{fs}}\right]}{\partial t} + \ \boldsymbol{\nabla}\cdot\left[c\phi\rho_{_{fs}}\dot{\mathbf{u}}_{_{fs}}\right] = \dot{m} \tag{3.12}
$$

where $\dot{\mathbf{u}}_{fs}$ is the absolute velocity of the fluidized solid phase.

fluid phase (ff)

Finally, the averaged mass of fluid automatically ensues as

$$M_{ff} = \int_{\Omega} (1-c)\,\phi\rho_{ff} d\Omega \tag{3.13}$$

with local mass balance for this phase being

$$\frac{\partial\left[(1-c)\,\phi\rho_{ff}\right]}{\partial t} + \boldsymbol{\nabla}\cdot\left[(1-c)\phi\rho_{ff}\dot{\mathbf{u}}_{ff}\right] = 0 \tag{3.14}$$

where ρ_{ff} = fluid density and $\dot{\mathbf{u}}_{ff}$ = absolute velocity of fluid phase.

In anticipation to the description of fluid flow through a porous medium, a volume averaged discharge velocity $\mathbf{v}_f$ of fluid mixture *relative to the solid matrix* is defined, i.e.

$$\mathbf{v}_f = \phi(\dot{\mathbf{u}}_{ff} - \dot{\mathbf{u}}_s) \tag{3.15}$$

Since $\dot{\mathbf{u}}_{ff} = \dot{\mathbf{u}}_{fs}$, and using Equation (3.15),

$$\dot{\mathbf{u}}_{ff} = \dot{\mathbf{u}}_{fs} = \mathbf{v}_f/\phi + \dot{\mathbf{u}}_s \tag{3.16}$$

Equations (3.10), (3.12), and (3.14) represent local mass balance equations for each individual phase. Successively combining Equations (3.10), (3.12), and (3.14) together, and eliminating fluidized solid and fluid velocities with the aid of Equation (3.15), the following three governing equations are obtained, i.e.

$$-\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\left[(1-\phi)\,\dot{\mathbf{u}}_s\right] = -\frac{\dot{m}}{\rho_s} \tag{3.17}$$

$$-\frac{\partial(1-c)\phi}{\partial t} + \boldsymbol{\nabla}\cdot\left[c\mathbf{v}_f + (1-\phi+\phi c)\,\dot{\mathbf{u}}_s\right] = 0 \tag{3.18}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v}_f = -\boldsymbol{\nabla} \cdot \dot{\mathbf{u}}_s = -\frac{\partial \varepsilon_V}{\partial t} \tag{3.19}$$

In Equation (3.19), due to incompressibility of the fluid, mass balance implies that the volume change of the matrix ε_v due to deformations corresponds to the net rate of change of fluid fluxes. Moreover, the porosity term in Equations (3.17) and (3.18) refers to total volume changes that arise from: (i) solid skeleton deformations as a result of grain rearrangements under stresses, and (ii) erosion as grains are dislodged from the matrix and enter the fluid phase. Thus, ideally the total porosity can be decomposed into stress induced (ϕ_σ) and erosion based (ϕ_{er}) porosities such that

$$\phi = \phi_\sigma + \phi_{er} \tag{3.20}$$

and

$$\frac{\partial \varepsilon_V}{\partial t} = \frac{\partial \phi_\sigma}{\partial t} \tag{3.21}$$

Equations (3.17-3.19) represent a system of 3 equations with 9 principal unknowns c, ϕ, $\dot{m}$, $\mathbf{v}_f$, and $\dot{\mathbf{u}}_s$ in the three dimensional case. More governing equations are needed in order to complete the description of the addressed physics.

3.3.2 Equilibrium and stress decomposition

The interaction between the mechanical behaviour of a deforming solid matrix and fluid dynamics must be incorporated into the governing equations in order to describe the coupling effects. For example, compression of the solid matrix leads to increased pore pressures under an undrained condition. Also, a change of pore pressure will induce a response of solid matrix, which may be unstable. When considering the deforming sand matrix under a total stress field $\boldsymbol{\sigma}$ in a quasi-static condition, the dynamic momentum equations are replaced by the corresponding overall static force equilibrium, i.e.

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{b} = 0 \tag{3.22}$$

where $\mathbf{b}$ are body forces per unit volume. In order to describe hydromechanical phenomena, i.e. the impact of interstitial fluids on deformations, the framework of effective stress in porous media is invoked in which the stress can be decomposed into a so-called effective stress $\boldsymbol{\sigma}'$ in the solid skeleton, and a fluid pressure component p in the pores in the case of full saturation with a pore fluid (e.g. water, oil) so that

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - \omega p\,\mathbf{1} \tag{3.23}$$

where ω is a parameter accounting for the compressibility of the solid grains. The sign convention adopted is negative stresses are compressive and fluid pressures are always positive, see Biot (1941[100], 1972[101]). Furthermore, compressibility coefficient ω can be written as

$$\omega = 1 - c_g/c_s \tag{3.24}$$

where c_g is the grain compressibility and c_s is the skeleton compressibility. If $c_s >> c_g$, then $\omega = 1$. It is instructive to have further insights into the nature of $\boldsymbol{\sigma}'$ which is defined as a measure of the intergranular forces and is based on stress averaging and decomposition, i.e.

$$\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}}_s(1 - \phi) - \phi p\mathbf{1} \tag{3.25}$$

where $\bar{\boldsymbol{\sigma}}_s$ is the average intergranular stress. By comparing Equation (3.23) with Equation (3.25), and assuming $\omega = 1$, the relation between effective stress and intergranular stresses emerges as

$$\boldsymbol{\sigma}' = (1 - \phi)\left(\bar{\boldsymbol{\sigma}}_s - p\mathbf{1}\right) \tag{3.26}$$

and it is the effective stresses that are used when calculating the constitutive response.

3.4 Constitutive laws

The solution of the boundary value problems can be achieved only after proper constitutive laws are determined, that is, these equations must be supplemented successively with an erosion law for characterizing mass generation or loss, Darcy's law for fluid flow, as well as stress and strain relationships for material strength.

3.4.1 Constitutive law for mass generation

An erosion phenomenon occurs when solid particles are lifted and agitated from the solid matrix by a viscous flow. It can be treated as a typically inverse problem of filtration. The filtration phenomenon refers to that the transport of fines or small solid particles by the fluid as it flows through a porous matrix or medium. During the filtration some of the fines are deposited in the upper layers of the filter, while the remaining portion is entrained to be deposited in deeper layers or removed out of the filter. It is believed that the deposition is caused by straining (fines deposit in the narrow spaces adjacent to contacts of matrix grains), settling (fines settle out by gravity and difference in floc and water densities), surface attractions, and bridging (extended form of straining, occurs in small pores) (Einstein, 1937[65]; Sakthivadival and Irmay, 1966[66]). This phenomenon has been studied extensively through both experimental and theoretical works that provide a basis for the current filtration theory. As such, Vardoulakis et al. (1996)[64] borrowed the concept and used it as a simple constitutive law to describe the rate of eroded sand particle mass in a fluid saturated sand under pressure gradient.

Intuitively, there must be a critical fluid velocity at which sand production is initiated. Once initiated, the rate at which solid grains are eroded depends principally on grain contact strength, local fluid drag forces, fluid pressures and availability of solids. Furthermore, it is clear that the erosion process is more intense in intact regions where porosity ϕ is small. Once the fluid velocity exceeds a critical value also based on strength, rate of erosion follows the fluidized solid velocity $c\mathbf{v}_f$. Thus, based on phenomenology and alluding to an inverse of the filtration theory, a possible functional form of mass generation can be written as

$$\begin{aligned} \frac{\dot{m}}{\rho_s} &= \lambda(1-\phi)c\left\|\mathbf{v}_f\right\| \qquad \text{if} \quad \|\mathbf{v}_f\| \geq \left\|\mathbf{v}_f^{cr}\right\| \\ &= 0 \qquad\qquad\qquad\quad \text{if} \quad \|\mathbf{v}_f\| < \left\|\mathbf{v}_f^{cr}\right\| \end{aligned} \tag{3.27}$$

where erosion coefficient λ, having the dimension of inverse of length, has to be determined experimentally. Basically, λ provides a length scale that can be related to the grain size and, most importantly, grain contact strength. In reference to the description of the stress/deformation aspects of the solid matrix (Equation 3.22),

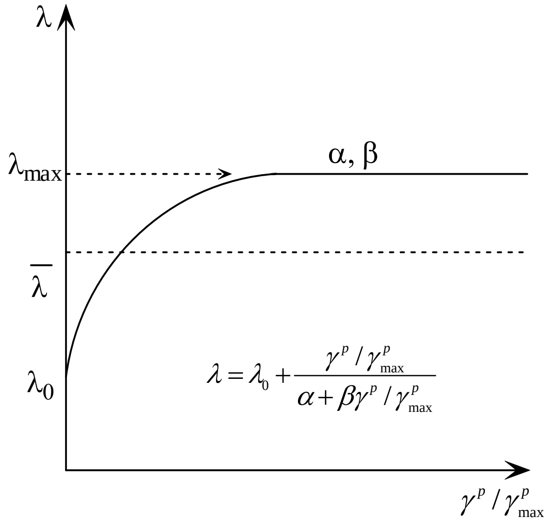


Figure 3.4: The relationship of erosion coefficient λ and plastic shear strain γ^p

the parameter λ can be linked to accumulated plastic shear strains γ^p through the following relationship, i.e.

$$\lambda = \lambda_0 + \frac{\gamma^p/\gamma^p_{\max}}{\alpha + \beta\gamma^p/\gamma^p_{\max}} \tag{3.28}$$

where α and β are constants, while λ_0 is the threshold strength at which erosion is triggered, and $\gamma^p_{\max}$ corresponds to the maximum plastic shear strain calculated for the entire domain. Equation (3.28) gives a hyperbolic variation of λ with respect to normalized plastic shear strains ($\gamma^p/\gamma^p_{\max}$), i.e. with increasing plastic strains, λ becomes larger which in turn increase the erosion activity as implied in Equation (3.27), see Figure 3.4.

Depending on the type of porous media, the corresponding constants α and β can be determined by experiments. In the case of suppressing the solid skeleton deformation, an equivalent erosion coefficient $\bar{\lambda}$ can be chosen as the mean value between λ_0 and $\lambda_{\max}$, such that reasonable erosion can take place. For instance, $\bar{\lambda}$ can be set to 2-10 for oil sands, 80-200 for sandstone as used in the forthcoming simulations. Furthermore, once the fluid velocity exceeds a critical value based on material strength, the rate of erosion follows the fluidized solid velocity $c \left\| \mathbf{v}_f \right\|$.

3.4.2 Constitutive law for fluid flow

When fluid is flowing into the well, the pressure and stress solutions will be time dependent. However, due to the complexity of flow in porous media, semi-empirical Darcy's law is used rather than the equations of fluid momentum balance. It is known that this law should be referred to as a "motion equation" for a fluid occupying the entire void space of a porous medium, which could be derived from Navier-Stokes equation (see details in Appendix A). Thus, a steady state condition is assumed as the Darcy's law is valid only for laminar flow corresponding to small Reynolds numbers (Re $<$ 1 to 10). Darcy's law expresses the proportionality of the specific flux or discharge (rate of flow per unit area) to the pressure gradient. It establishes the relationship between pressure gradient $\boldsymbol{\nabla} p$ and volumetric fluid mixture flux per unit area, $\mathbf{v}_f$. Thus,

$$\mathbf{v}_f = -\frac{\mathbf{k}}{\mu} \cdot \boldsymbol{\nabla} p \tag{3.29}$$

where $\mathbf{k}$ is the effective permeability tensor that can be related to porosity via Carman-Kozeny equation (Carman, 1956)[102], i.e.

$$\mathbf{k} = k_0 \frac{\phi^3}{(1-\phi)^2}\,\mathbf{1} \tag{3.30}$$

and k_0 is a constant for a given sand matrix that depends upon the grain size, shape and their arrangement (structure, packing). The Kronecker delta tensor is $\mathbf{1}$ such that $\mathbf{1}_{ij} = \delta_{ij}$. Alternatively, a variant of Carman-Kozeny equation may be used instead as far as the computational considerations, i.e.

$$\mathbf{k} = k_0 \exp(A_0 \frac{1-\phi}{1-\phi_0})\,\mathbf{1} \tag{3.31}$$

and k_0 and ϕ_0 are constants, and A_0 is a fitting parameter.

Equations (3.30) and (3.31) describe how the permeability changes in the eroded regions as a function of porosity. Furthermore, the parameter μ in Equation (3.29) refers to the viscosity of the fluidized sand and fluid mixture. It can be related to the kinematic viscosity η of the fluid using averaging of the mixture of phases, i.e.

$$\mu = \bar{\rho}(c)\eta; \qquad \bar{\rho}(c) = (1-c)\rho_f + c\rho_s \tag{3.32}$$

in which $\bar{\rho}(c)$ is the density of the fluidized sand and fluid mixture. Based on typical densities of fluid and solid, the viscosity of mixture can range from the value of the fluid viscosity at the beginning, to up to three times in later times.

3.4.3 Constitutive law for porous media

Turning to solid skeleton deformations, an integrated constitutive law (stress-strain relationship) must be introduced. Sand is one of the typical examples of granular materials that exhibit predominantly nonlinear stress-strain behaviour upon loading as well as unloading and reloading. The pressure sensitivity due to the existence of internal friction is the key property which differentiates itself from other solids, i.e. metals. Hence, constitutive laws for granular materials depend on current external and internal state variables such as stress, strain, geometric fabric, density, as well as deformation history. Comprehensive studies on these laws are outside the scope of this research. In this section, only classical elastic and elasto-plastic constitutive laws are reviewed and implemented in this thesis.

Poroelasticity

The real behaviour of porous media is not linear elastic, but linear elastic analysis may be a good starting point towards understanding the response of porous media under complicated consideration of constitutive laws. In linear isotropic poroelasticity, the effective stress tensor is related to the strain tensor through a fourth order elastic tensor, $\mathbf{C}^e$, with two parameters, i.e. elastic modulus E and Poisson ratio υ. Thus,

$$\boldsymbol{\sigma}' = \mathbf{C}^e \cdot \boldsymbol{\varepsilon} \tag{3.33}$$

where

$$\mathbf{C}^e = K\,\mathbf{1} \otimes \mathbf{1} + 2G^e \left(\mathbf{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1} \right) \tag{3.34}$$

with

$$\mathbf{I}_{ijkl} = \frac{1}{2}(\delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl}) \tag{3.35}$$

The symbol $\otimes$ represents the tensor product operator in the sense that $(\mathbf{1} \otimes \mathbf{1})_{ijkl}$ implies $\delta_{ij}\delta_{kl}$ (δ_{ij} : Kronecker delta). Elastic moduli K and G^e are bulk and shear moduli expressed in terms of E and υ, i.e.

$$K = \frac{E}{3\,(1-2\upsilon)}; \qquad G^e = \frac{E}{2(1+\upsilon)} \tag{3.36}$$

An improvement over simple elasticity can be made by considering nonlinear elasticity with a variable elastic modulus $E^{'}$. Equation (3.26) suggests that the effective stresses borne by the solid phase in dry conditions is $\boldsymbol{\sigma'} = (1-\phi)\bar{\boldsymbol{\sigma}}_s$. Thus, this means that effective elastic modulus E' takes the form of

$$E' = E\,(1-\phi) \tag{3.37}$$

with ϕ acting as a damage parameter applied to the elastic modulus E of the solid material.

Poroplasticity

The assumption of simple elastic behaviour for the solid phase can be lifted, and thereby replaced with a constitutive theory which considers the dissipative nature of sand behaviour resulting from micro-processes such as grain slippage, rearrangement, dilation and destructuration. As the sand is sheared under the action of drilling for example, an increase in volume (dilation) has to occur for deformations to mobilize. This behavioural aspect has indeed an impact on sand production, and has to be addressed within elasto-plasticity theory which invokes a yield criterion combined with a plastic flow rule to describe yield condition and plastic strains respectively.

Hence, a more adequate constitutive law based on plasticity and incorporating stress dilatancy aspects must be used.

The plasticity emerges as the irrecoverable strains that are produced upon load removal. Thus, both loading and unloading have to be treated differently to characterize the non-unique stress-strain relation as opposed to that of non-linear elasticity. In order to formulate a theory which models elasto-plastic material deformation three requirements have to be met (Owen and Hinton, 1980[103]):

- An explicit stress-strain relationship must be formulated under elastic conditions (before the onset of plastic deformation).
- A yield criterion must be postulated to indicate the stress level at which plastic flow occurs.
- A stress-strain relationship must be developed for post-yield behaviour.

After initial yielding the material behaviour is partly elastic and partly plastic. In order to be capable of taking into account the dependence of plastic behaviour on loading history, an incremental stress-strain approach is used. Hence, the total strain increments are assumed to be divisible into elastic and plastic components,

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p \tag{3.38}$$

The elastic strain increments are given by

$$d\boldsymbol{\varepsilon}^e = \mathbf{D}^{-1} \cdot d\boldsymbol{\sigma}' \tag{3.39}$$

where $\mathbf{D}^{-1} = \mathbf{C}^e$, which is defined in Equation (3.34).

The plastic strain is irreversible, not time dependent, and can only be sustained once a certain level of stress has been reached. In order to derive the relationship between the plastic strain component and the stress increment, a flow rule must be assumed to estimate the plastic strain increments during yielding. In particular, the plastic strain is assumed to be proportional to the effective stress gradient of the plastic potential G. Thus, the plastic strain increment is given by

$$d\boldsymbol{\varepsilon}^p = d\Lambda \frac{\partial G}{\partial \boldsymbol{\sigma}'} \tag{3.40}$$

where $d\Lambda$ is a proportionality constant termed the plastic multiplier, as yet undetermined. Such flow rule determines the direction and the amplitude of the plastic strain. Thus, combining Equations (3.38) with (3.40), the total strain increments become,

$$d\boldsymbol{\varepsilon} = \mathbf{D}^{-1} \cdot d\boldsymbol{\sigma}' + d\Lambda \frac{\partial G}{\partial \boldsymbol{\sigma}'} \tag{3.41}$$

The plastic increment of strain will occur if the elastic stress increment tends to put the stress outside the yield surface. The plastic multiplier $d\Lambda$ is obtained from consistency condition, i.e.

$$F(\boldsymbol{\sigma}', \kappa) = 0 \tag{3.42}$$

where κ is a hardening parameter. Differentiating Equation (3.42) ultimately gives,

$$\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}'}\right\}^T \cdot d\boldsymbol{\sigma}' - A\ d\Lambda = 0 \tag{3.43}$$

where

$$A = -\frac{\partial F}{\partial \kappa}\frac{\partial \kappa}{\partial \Lambda} \tag{3.44}$$

where A is a scalar term that can be determined according to the hypothesis of hardening rules such as work hardening and strain hardening. For example, if assuming the work hardening, A is obtained to be the local slope of the uniaxial stress/plastic strain curve and can be determined experimentally. Clearly, A is simply zero for ideal plasticity with no hardening.

Equations (3.41) and (3.43) can now be written in a single matrix form as

$$\left\{\begin{array}{c} d\boldsymbol{\varepsilon} \\ 0 \end{array}\right\} = \left[\begin{array}{cc} \mathbf{D}^{-1} & \frac{\partial G}{\partial \boldsymbol{\sigma}'} \\ \left\{\frac{\partial F}{\partial \boldsymbol{\sigma}'}\right\} & -A \end{array}\right] \left\{\begin{array}{c} d\boldsymbol{\sigma}' \\ d\Lambda \end{array}\right\} \tag{3.45}$$

Thus, Equation (3.38) can be expressed as

$$d\boldsymbol{\sigma}' = \mathbf{D}_{ep} \cdot d\boldsymbol{\varepsilon} \tag{3.46}$$

where elasto-plastic matrix

$$\mathbf{D}_{ep} = \mathbf{D} - \frac{\mathbf{D}\cdot \left\{\frac{\partial G}{\partial \boldsymbol{\sigma}'}\right\} \cdot \left\{\frac{\partial F}{\partial \boldsymbol{\sigma}'}\right\}^T \cdot \mathbf{D}}{A + \left\{\frac{\partial F}{\partial \boldsymbol{\sigma}'}\right\}^T \cdot \mathbf{D}\cdot \left\{\frac{\partial G}{\partial \boldsymbol{\sigma}'}\right\}} \tag{3.47}$$

As long as the yield function, potential function and the hardening rules are given, the stress increments are well defined. Particularly, if $G = F$, an associated plasticity rule is obtained, while $G \neq F$ refers to non-associated plasticity.

In order to properly address the sand production problem, a yield function $F(\boldsymbol{\sigma}\prime)$ based on Mohr-Coulomb is considered adequate, while plastic strains are calculated using an appropriate form of plastic potential function G derived by Wan and Guo (1998)[104] based on stress-dilatancy theory. In particular, the flow rule basically defines the plastic strain increment vector as the normal to the plastic potential function G and its magnitude determined from the plastic multiplier Λ, i.e.

$$\begin{aligned} d\boldsymbol{\varepsilon}^p &= d\Lambda \frac{\partial G}{\partial \boldsymbol{\sigma}'} \\ d\Lambda &\geq 0 \text{ if } F(\boldsymbol{\sigma}') = 0 \ \text{ and } \ dF = 0 \\ d\Lambda &= 0 \text{ if } F(\boldsymbol{\sigma}') = 0 \ \text{ and } \ dF < 0 \end{aligned} \tag{3.48}$$

The main ingredients are covered in details in Wan and Guo (1998[104], 1999[45]).

3.5 Initial and boundary conditions

These above-mentioned governing equations (Equations 3.17, 3.18, 3.19, and 3.22) supplying the constitutive equations (Equations 3.27, 3.29, 3.33, and 3.38) can only be solved with the associated initial and boundary conditions involving unknown variables such as c, ϕ , p, and $\mathbf{u}_s$ lumped together into the generic vector $\mathbf{W}$. As such, typical initial boundary conditions can be expressed as

$$\mathbf{W}(\mathbf{X},\ t = 0) = \mathbf{W}_0 \ \text{ in } \Omega \tag{3.49}$$

where $\mathbf{X}$ is the position in the problem domain, and $\mathbf{W}_0$ contains the initial values.

Turning to boundary conditions, they can be typically of the Dirichlet- or Neumann-type. The Dirichlet-type condition corresponds to variables being prescribed to values $\bar{\mathbf{W}}$, i.e.

$$\mathbf{W} - \bar{\mathbf{W}} = \mathbf{0} \quad \text{on } \Gamma_{\bar{W}} \tag{3.50}$$

where $\Gamma_{\bar{\mathbf{W}}}$ is the Dirichlet boundary. On the other hand, Neumann-type condition refers to flux type boundary conditions where

$$\boldsymbol{\nabla}_n \cdot \mathbf{W} - \boldsymbol{\nabla}_n \cdot \bar{\mathbf{W}} = \mathbf{0} \qquad \text{on } \Gamma_n \tag{3.51}$$

with Γ_n being the Neumann boundary and n the normal to it.

3.6 Summary and discussions

A framework has been proposed in which coupled stress-deformation-erosion phenomena can be modelled to describe the sand production problem by linking skeleton volume changes (i.e. porosity) to fluid transport in a consistent manner via theory of mixtures. Mass balance equations are written in a consistent manner to couple the behaviour of fluid and solid skeleton by considering an erosion process in a deformable solid matrix. A simple empirical erosion law has been used to describe source terms in mass balance equations to account for the rate of mass loss or gain between solid and fluidized solid phases. Semi-empirical Darcy's law has been used to link the fluid velocity to the pressure gradient via a porosity-dependent permeability (i.e. Carman-Kozeny Equation 3.30). A generalized stress-strain relationship has been sought to characterize the behaviour of the sand matrix. Even though, the model presented in this chapter is based on single phase flow, it provides a general framework upon which more features can be added so as to account for the effect of a gas phase.

Once constitutive laws that define the material properties are introduced, the solution of the boundary value problem can be initiated with the proper boundary condition and initial conditions being applied. Usually, an advanced numerical modelling technique such as finite element method is needed to seek for the solution of principal field unknowns such as concentration c, porosity ϕ, pressure p, and displacements of the porous matrix $\mathbf{u}_s$. Within the context of initial and boundary condition settings, the coupled time-dependent boundary value problem can only be solved numerically using finite element discretization along with an advanced iterative scheme such as the Newton-Raphson scheme in order to deal with the strong non-linearity. In the next chapter, the governing equations will be discretized using standard Galerkin's method in view of computing sand production in an initial and boundary value problem setting.